

Guinier's Law

Beginning with the Amplitude equation

$$A(q) = \int \rho(r) e^{-iqr} dV$$

expand the exponential

$$A(q) = \int \rho(r) dV - i \int qr \rho(r) dV - \frac{1}{2!} \int (qr)^2 \rho(r) dV + \dots$$

given $\int \rho(r) dV = \rho_0 V$ and $\int r \rho(r) dV = 0$ when the origin of r is at the center of mass (i.e. $\sum_{i=1}^n r_i = r_1 + r_2 - r_3 - r_4 \dots = 0$ if r_i is centered at 0) of the particle then:

$$A(q) = \rho_0 V - 0 - \frac{1}{2} (qr)^2 \int \rho(r) dV + \dots$$

$$A(q) = \rho_0 V \left[1 - \frac{1}{2} (qr)^2 \dots \right]$$

The intensity is obtained by squaring the amplitude so :

$$I(q) = \langle A(q) \rangle^2 = \rho_0^2 V^2 \left[1 - \frac{1}{2} q^2 r^2 \dots \right]^2$$

$$I(q) = \rho_0^2 V^2 \left[1 - q^2 r^2 + \frac{1}{4} q^4 r^4 \dots \right]$$

For small q then $q^4 r^4 \ll \ll q^2 r^2$ and the approximation

$$I(q) = \rho_0^2 V^2 \left[1 - q^2 r^2 \dots \right] \text{ is valid.}$$

r is a vector that can be equated to the Cartesian coordinate system as

$$r^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2 + 2\bar{x}\bar{y} + 2\bar{x}\bar{z} + 2\bar{y}\bar{z}$$

and

$$q^2 r^2 = q_x^2 \bar{x}^2 + q_y^2 \bar{y}^2 + q_z^2 \bar{z}^2 + 2q_x^2 q_y^2 \bar{x}\bar{y} + 2q_x^2 q_z^2 \bar{x}\bar{z} + 2q_y^2 q_z^2 \bar{y}\bar{z}$$

For an isotropic system $\bar{x}^2 = \bar{y}^2 = \bar{z}^2 = \frac{1}{3} R_g^2$, $\bar{x}\bar{y} = \bar{x}\bar{z} = \bar{y}\bar{z} = 0$ and

$q_x^2 = q_y^2 = q_z^2 = q^2$ therefore

$$I(q) \cong \rho_0^2 V^2 \left[1 - q^2 \frac{1}{3} R_g^2 \dots \right]$$

or

$$I(q) \cong \rho_0^2 V^2 \exp\left(-\frac{1}{3} q^2 R_g^2\right)$$

Which is known as Guinier's Law (approximation).